Rivalry Network

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Contents

1 Introduction

Rivalry network was originally proposed to study the behavior of coupled neuron populations. The original Wilson-Cowan model focuses on the rivalry network between two cells or two cell populations. The extended model, which studies more interconnected cells, has been proved useful in explaining some periodical biology phenomena. In this paper, I will particularly focus 2 of them: visual illusion and animal gaits. I will explain the original model and its periodical behavior, and with symmetry, I will show how in the extended model the periodical behavior explains alternating and fusion.

Animal gaits are behaviors controlled by the central pattern generator (CPG) that generating rhythmic outputs. (Martin et al., 1999, p. 693). In quadrupedal gaits, the rhythms of walk, tort, pace and etc. lead to the assumption of a symmetric 8-cell rivalry network model. The model, which is the extended Wilson-Cowan model, is able to generate rhythmic patterns observed in quadrupedal locomotion. (p. 694). These patterns are determined by symmetric analysis.

Visual illusion is a general description for a set of various puzzling visual phenomena, including binocular rivalry, multi-stable figures, and impossible objects. Binocular rivalry occurs when 2 different images presented simultaneously to each eye. Typical features of binocular rivalry are alternating or mixing perceptions of the presented images. Illusion describes the alternation in possible perceptions of 1 presented image. Impossible objects occurs when some locally consistent parts are combined in some ways that are not globally consistent. This paper will mainly focusing on explaining binocular rivalry with extended Wilson-Cowan model.

2 Literature Review

2.1 Wilson Network

In the attempt of explaining higher level sensory functions, Hugh R. Wilson and Jack D. Cowan (1972) proposed that the study method that focuses on the genetically determined features of individual neurons and their interactions with other cells may not be appropriate. The pragmatical concern is that cells involved in a particular sensory function is "too vast for any approach starting from single cells to be tractable" (Wilson & Cowan, 1972, p.1). Therefore, they proposed a generalized rivalry network between cells populations of excitatory or inhibitory activities. The original network consist of only 2 nodes, one excitatory and one inhibitory. The behavior of this model depends on various parameters in the model:

the strength external excitatory or inhibitory stimulus, the strength of the excitatory or inhibitory connections between nodes, and the strength of self-stimulus in a node. With some specific combinations of parameter values, one stable node and one saddle not, one stable and one unstable node, or one limit cycle and 2 saddle nodes behaviors can be achieved in this model. Further, Wilson proposed a rivalry network that describes rivalry between generalized patterns, which are described by a collection of attributes. The generalized rivalry network can be described by a matrix whose columns represent attributes and rows represent levels of attribute. A pattern is then a choice of one level for each attribute. An simple example is shown in Figure 2.1.1.

Such a generalized rivalry network is called a Wilson network. The models that will be introduced in this paper can all be viewed as special examples of Wilson networks. A Wilson network is specified with two types of connections. The first is all-to-all inhibitory connections in each column. The second is the specific excitatory connections between different columns and levels.

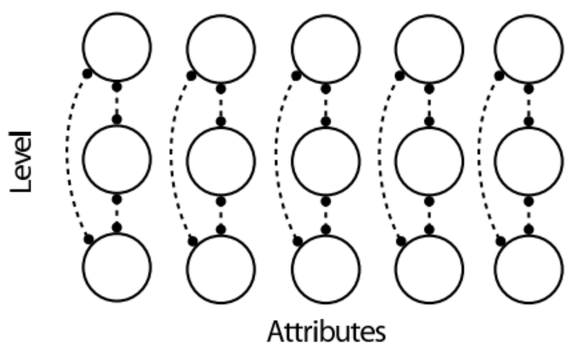


Figure 2.1.1 Example of a Wilson network, (Lan & Martin, 2022, unpublished)

2.2 Binocular Rivalry

The binocular rivalry was discoevered by Giambattista della Porta in 1593. He presented to himself 2 books to each eye and found that he was able to read from either book. A typical modern experiment of binocular rivalry is demonstrated in Figure 2.2.1. The images will be presented to each eye of the subject. In such experiments, subjects usually report perceiving one image and then the other, and the alternation never stops. Such alternation can be explained by the periodical solution, or limit cycle behavior, of the Wilson-Cowan model.

However, the 2 node Wilson-Cowan model is not able to explain all binocular rivalry phenomena. In 1996, Kovács, Papathomas, Yang, and Fehér published the "Monkey-Text" experiment. The first experiment is similar to the previous one: the subjects was presented a picture of a monkey to one eye and a jungle scene with text to the other, as described in Figure 2.2.2. Without surprise, the reported result is identical with the previous experiment. Subjects report perceiving alternating image between monkey and text. In the second experiment, the images are subdivided into several pieces and shuffled, as presented in Figure 2.2.3. This time, subjects reports 4 different perceptions: Mixed monkey, mixed text, Monkey, and text. Clearly, the original 2 node Wilson-Cowan model is not enough to explain the alternation of 4 different perceptions. Therefore, a 4 node generalized Wilson network is proposed to explain. How could a 4 node generalized Wilson network produce 4 alternating results will be explained in later sections of this paper. The "Monkey-text" experiment can be simplified the 24-dot experiment, as presented in Figure 2.2.4. These experiments can be explained by the result of a 4 node generalized rivalry network. The analytical results of the further extended 8 node has not been verified by directly and exactly by any known experiment. In the analysis of these generalized rivalry work, symmetry of the network helps determine the patterns of interocular grouping (Golubisky et al., 2019, p. 1989).

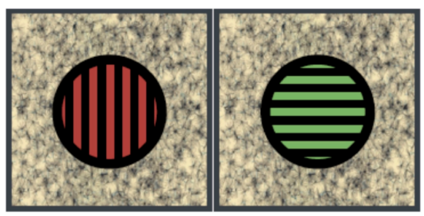


Figure 2.2.1 A typical binocular rivalry experiment, (Lan & Martin, 2022, p. 643)

Figure 2.2.2 First Monkey-text experiment, (Lan & Martin, 2022, p. 649)

Figure 2.2.3 First Monkey-text experiment, (Lan & Martin, 2022, p. 649)

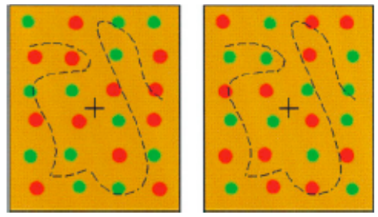


Figure 2.2.4 24dots experiment, (Lan & Martin, 2022, p. 652)

2.3 Quadrupedal Gaits

Quadrupedal gaits have been studied in various ways including: equivariant bifurcation theory (Collins & Stewart, 1993), numerical simulations (Collins & Richmond, 1994), and phase response curves (Canavier et al., 1997). Analysis in this paper is based on the 8-cell rivalry network model introduced by Golubitsky et al. (1998). This model is able to produce periodic solutions that simulates the output signals from CPG. As the result, the quadrupedal gaits predicted by this model is the primary gaits, which are walk, trot, pace, bound, jump, and pronk. The 2 main contributions of this model is following. First, this model is the smallest one under the same assumptions. Second, any primary gaits, except pronk, can be produced by Hopf bifurcation by varying only coupling strength between cells (Golubitsky et al, 2001).

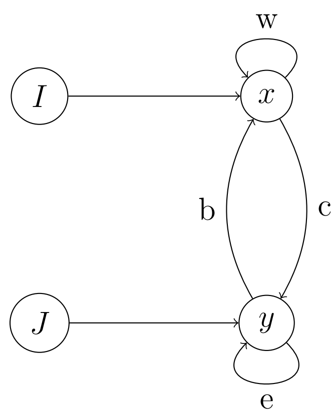
3 Methods and Results

3.1 Wilson-Cowan Model

The originally proposed 2-node Wilson-Cowan model can be demonstrated by Graph 3.1.1. Here $x(t)$ and $y(t)$ are 2 nodes of opposite patterns (which is excitatory and inhibitory in this context). I and J are the external stimulus. b, c are the connections between two nodes, which are inhibitory to the activity of each of them (i.e. for the excitatory node the signal received is inhibitory and for inhibitory node the signal received is excitatory; both will reduce the strength of activity in the targeted node). The model equation following this demonstration will have the form:

$$\begin{aligned} \tfrac{dx(t)}{dt} &= -ax(t) + \mathcal{G}(wx(t) - by(t) + I(t)) \\ \tfrac{dy(t)}{dt} &= -dy(t) + \mathcal{G}(cx(t) - ey(t) + J(t)) \end{aligned} \tag{1}$$

Here a, d are the natural decay rate of the signal; w, e are the strength of self stimulation; c, b are the strength of connection between nodes. $\mathcal{G}$ is any sigmoid gain function models the effect of all inputs to the node. $\mathcal{G}$ has to have the following properties: (1) $\mathcal{G}(z)$ is monotonically increasing; (2) When $z \to \infty$, $\mathcal{G}(z) \to C_1$, and when $z \to -\infty$, $\mathcal{G}(z) \to C_2$, $C_1, C_2 \in \mathbb{R}$; (3) $\mathcal{G}$ has and only has 1 inflection point.



Graph 3.1.1. The 2-node Wilson-Cowan Model

Behavior of this model can be studied with numerical simulations and explained with Hopf bifurcation.For almost-linear systems, the occurrence of Hopf bifurcation can be determined by the Jacobian matrix of system, which is:

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \tag{2}$$

for a dynamical system $\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$. The system is almost linear (i.e. it can be examined with the Jacobian matrix) if it has continuous first partial derivatives. Denote the determinant of the matrix as D and the trace as T. Hopf bifurcation occurs when the sign of T changes sign while D is always positive. The process can be demonstrated with Graph 3.1.2., where different regions denotes the corresponding behavior of the model with specific T, D, and Hopf bifurcation occurs (locally) when T changes between the blue and yellow region.

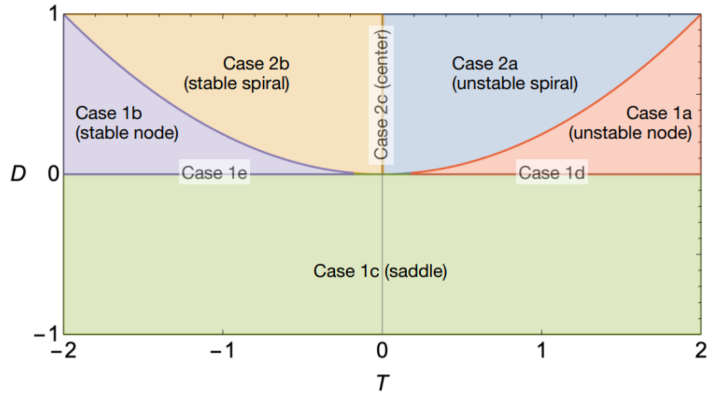


Figure 3.1.2. Regions of Behaviors, (Konstatinos Efstathiou, 2021, p. 27)

With this result, we can further analyze the behavior of this model with numerical analysis. Since S can be any function satisfying those three conditions listed above, we can assume $S(z) = \frac{z}{\sqrt{z^2+1}}$ without loss of generality.

Then the system becomes:

$$\begin{aligned} \frac{d(x)}{dt} &= f(x,y) = -ax + \frac{wx-by+I}{\sqrt{(wx-by+I)^2+1}} \\ \frac{d(y)}{dt} &= g(x,y) = -dy + \frac{cx-ey+J}{\sqrt{(cx-ey+J)^2+1}} \end{aligned} \tag{3}$$

For this system, presented below is how its behavior changes with the changes of the external stimulation I when other parameters are fixed and $J = I$(which is useful in our following analysis). This simulation is done by Mathematica with code in Listing 1 in Appendix. The axes are the x and y axis. Lines labeled with arrows are the vector fields of the system. The red solid line is the trajectory starting the small dot from $t = 0$ to $t = 1000$. How changes of other parameters can affect the behavior of the model can be done similarly.

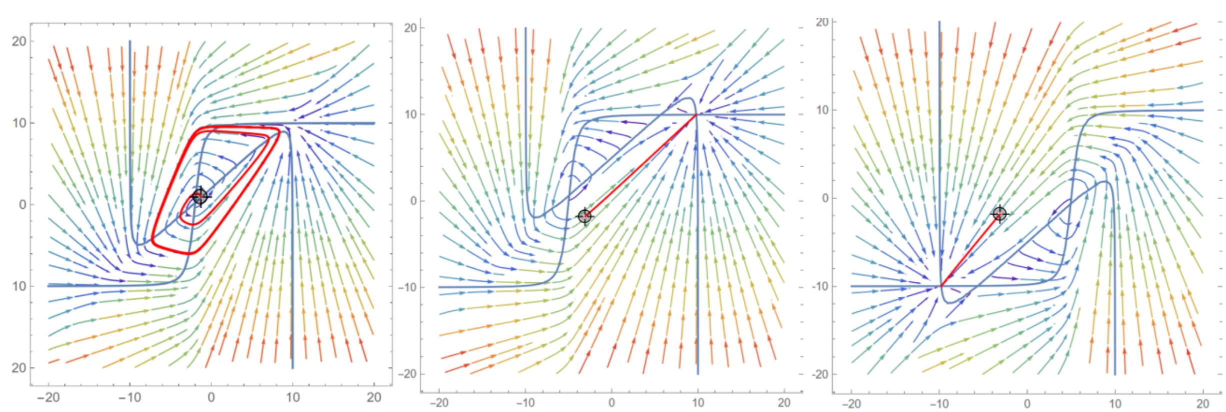


Figure 3.1.3. Simulations of the model when $a = b = 0.1, b = c = w = 1, e = 0, I = J = 2, 5, -5$

As shown in Figure 3.1.3. set $a = b = 0.1, b = c = w = 1, e = 0$, and $I = J$. Then when $I = J = 2$, there is a limit cycle around the origin; when $I, J = 5$ or -5, there is a stable node.

3.2 1-location Binocular Rivalry

The simplest binocular rivalry consists of two incompatible stimulus presented to the only retina location in each eye, as shown in Figure 3.2.1.

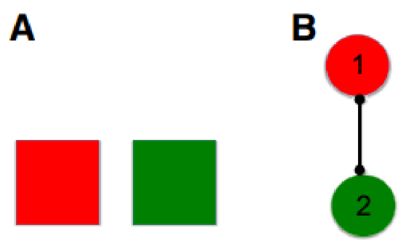


Figure 3.2.1. 1-location Binocular Rivalry: **A**-Two squares in red and in green of identical size, presented to each eye of subjects; **B**: Wilson network with 1 attribute and 2 levels. Symmetry group is $Z_2(\rho)$ (Golubitsky et al., 2019, p. 1990)

Since there are only 2 node involved in the model, we can fit the Wilson-Cowan model directly in the model.As only one retina location is assumed, only one attribute is included. Rivalry between 2 levels in this attribute corresponds to the inhibitory connections between 2 nodes receiving different simulations. For the nature of this experiment, it is safe to assume that the strength of external stimulus presented to each eye is equal (i.e. $I = J$). In the context of the experiment, when $x > y$, the stimulus presented to the left eye is perceived and when $y > x$, the stimulus presented to the right eye is perceived.

With the analysis of Wilson-Cowan model above, we can find that all behaviors of the model are periodical, invariant of the I we choose. It is either a limit cycle (which is clearly periodical: the whole solution goes back to certain point on the circle after some time) or a node ($x = y$ no matter what t we choose after some time).

Therefore, when we apply a $\mathbf{Z}_2$ (about this, see **Explanation 1: Symmetry** in Appendix) symmetry operator $\rho(12)$, we will have the solution of operated solution on the trajectory of the original solution (Denoted as $X(x(t), y(t))$):

$$\rho X(t) = X(t + \theta T) \tag{4}$$

The solution will not change as symmetry will not change types connections. What is done in the process is simply arranging parameters to the different variable. Therefore, after applying $\rho(12)$ twice we can obtain:

$$X(t) = X(t + 2\theta T) \tag{5}$$

Then, $\theta = \frac{1}{2}$ or $\theta = 0$. The second case is simply fusion, where by our rule, no clear result can be perceived. The first case implies $y(t) = x(t + \frac{T}{2})$ for all t. This further implies the alternation between 2 different perceptions, as $y(t) > x(t) \implies x(t + \frac{T}{2}) > y(t + \frac{T}{2})$

One remark here is that phase shift associated with symmetries are rigid (Golubitsky & Stewart, 2002). This indicated small changes in parameters will perturb the solution but will not change phase shift (since no parameter participates in the symmetry analysis above).

3.3 2-location Binocular Rivalry

As described in the literature review, while giving insights into the simplest binocular rivalry experiment, it is not able to generate solutions that explains more complicated experimental results, like the "monkey-text" experiment. In this concern, it is necessary to introduce binocular rivalry with more than one retinal locations. In this part, I will focus on the analysis of binocular rivalry with 2 retinal locations. As the retinal location increases to 2, in the corresponding Wilson network, another attribute column should be added, as demonstrated following in Graph 3.3.1:

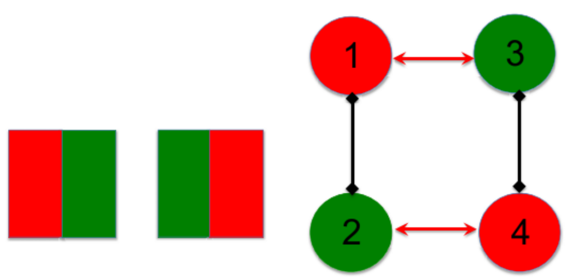


Figure 3.3.1. 1-location Binocular Rivalry: **A**-4 rectangles in red and in green of identical size, presented to each eye of subjects as pairs of 2; **B**: Wilson network with 2 attributes and 2 levels. Symmetry group is $Z_2(\rho) \times Z_2(\kappa)$ (Golubitsky et al., 2019, p. 1990).

Denote the strength of signals at point $1, 2, 3, 4$ as X_1, X_2, X_3, X_4. Then the system can be written as follows:

$$\begin{aligned} \dot{X_1} &= F(X_1, X_2, X_3, I) \\ \dot{X_2} &= F(X_2, X_1, X_4, I) \\ \dot{X_3} &= F(X_3, X_4, X_1, I) \\ \dot{X_4} &= F(X_4, X_2, X_3, I) \end{aligned} \tag{6}$$

With similar rules, perception states can be described as follows:

$$\begin{array}{ll} RG & X_1 > X_2, X_3 > X_4 \\ RR & X_1 > X_2, X_3 < X_4 \\ GG & X_1 < X_2, X_3 > X_4 \\ GR & X_1 < X_2, X_3 < X_4 \end{array} \tag{7}$$

This time, the symmetry operations can be applied are $\rho(12)(34)$ and $\kappa(13)(24)$. Similar to the previous case, $X(t) = (X_1, X_2, X_3, X_4)$ is periodic. If $\rho X(t) = X(t)$, then $X_1 = X_2$ and $X_3 = X_4$. This implies fusion in both attributes, in which case no alternation can happen. If $\rho X(t) = X(t + \frac{T}{2})$, the perceptual alternation occurs. Based on this, if $\kappa X(t) = X(t)$, $X_1 = X_3, X_2 = X_4$. According to the rule, the perception should alternate between RG and GR. Similarly, if $\kappa X(t) = X(t + \frac{T}{2})$, $X_1 > X_2 \implies X_3 < X_4$ and vise versa. Therefore, the perception alternates between RR and GG according to the rule.

This analysis explains the experimental results of the monkey-text experiment, where subjects report perceiving both mixed pictures and unmixed picture when presented with mixed pictures.

3.4 4-location Binocular Rivalry

Further, we extend the number of retinal locations to 4. This time we do not have a known experiment data to refer to. An experiment designed to verify the results of symmetry analysis will be introduced in the following subsections.

Before performing symmetry analysis about this model, configurations of input signals should be first discussed. Configuration is not discussed in the previous subsection as there are only two possibilities: pure color or mixed color (presented to each eye). For pure color, the model will be no different from the 1-location binocular rivalry model. Therefore, it is reasonable to discuss only 1 configuration. However, for this context, there are in total 6 configurations, 5 of which are different from the pure color model. In this case, it will be necessary to discuss them.

All 6 possible configurations are presented in Figure 3.4.1.In each configuration, the left image is the combination of color squares presented to the left eye and the right image is A and B are the same configurations as C and D but with no spacing between blocks. They will not be discussed in this subsection. This is set for the later experiment to study how spacing can influence perceptions.

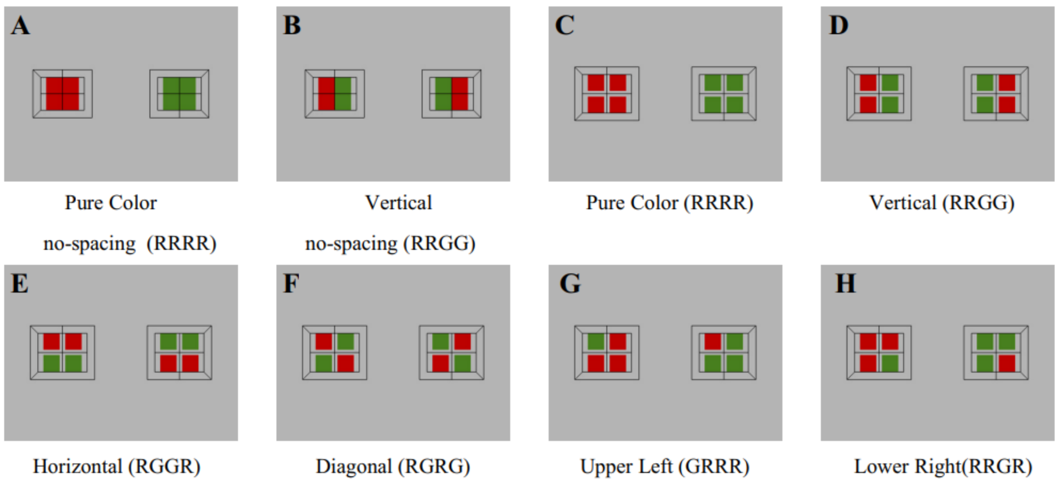


Figure 3.4.1. 1-location Binocular Rivalry: **A**-4 rectangles in red and in green of identical size, presented to each eye of subjects as pairs of 2; **B**: Wilson network with 2 attributes and 2 levels. (Golubitsky et al., 2019, p. 1992).

Similar to previous cases, when the retinal locations increase to 4, the corresponding Wilson network has 4 attributes, each of which consists of 2 levels with competing signals. Beside the spatial feature, with color feature taken into consideration (i.e connections between same colors and different colors in the original configuration is viewed as different. Some symmetry operation cannot be applied as a result), the following three symmetry operation could be applied to configuration C, D, E, F.

$$\begin{aligned} \rho &= (12)(34)(56)(78) \\ \tau &= (17)(28)(35)(46) \\ \kappa &= (13)(24)(57)(68) \end{aligned} \tag{8}$$

Similar to the previous case, in configuration C, D, E, F, denote $X(t) = (X_1, X_2, \ldots, X_8)$. $\rho X(t) = X(t)$ indicates the fusion of 2 levels in each attributes. Therefore, no visual alternation could occur. When ρ is not trivial (i.e. $\rho X(t) = X(t + \frac{T}{2})$, the notation holds for discussion about τ and κ), whether τ, κ are trivial or not leads to the following 4 phase shift patterns:

- α: τ, κ are both trivial $\implies X(t) = (X_1(t), X_1(t+\frac{T}{2}), X_1(t), X_1(t+\frac{T}{2}), X_1(t), X_1(t+\frac{T}{2}), X_1(t), X_1(t+\frac{T}{2})$
- β: τ is trivial but κ is nontrivial $\implies X(t) = (X_1(t), X_1(t+\frac{T}{2}), X_1(t+\frac{T}{2}), X_1(t), X_1(t+\frac{T}{2}), X_1(t), X_1(t), X_1(t+\frac{T}{2}))$
- γ: τ is not trivial but κ is trivial $\implies X(t) = (X_1(t), X_1(t+\frac{T}{2}), X_1(t), X_1(t+\frac{T}{2}), X_1(t+\frac{T}{2}), X_1(t), X_1(t+\frac{T}{2}), X_1(t))$
- δ: τ, κ are both nontrivial $\implies X(t) = (X_1(t), X_1(t+\frac{T}{2}), X_1(t+\frac{T}{2}), X_1(t), X_1(t), X_1(t+\frac{T}{2}), X_1(t+\frac{T}{2}), X_1(t))$

Fit the alternating pattern generated by these phase shift patterns into the rules determined by the original configurations and we can obtain the alternating pattern for each configuration from C to F.

For configuration G and H, only ρ can be applied as τ and κ will change the properties of connections. For example (see Graph 3.4.2), applying τ to configuration H, which will switch the location of 1 and 7, 3 and 5 in one level (one eye), replaces the connections between 1 and 3 with the connections between 7 and 5. Then the connection becomes a connection between different colors, while it is originally the connection between the same color. With the single symmetry operation ρ applied, non-fusion perceptions are obtained when $\rho X(t) = X(t+\frac{T}{2})$, which gives $X(t) = (X_1(t), X_1(t+\frac{T}{2}), X_3(t), X_3(t+\frac{T}{2}), X_5(t), X)5(t+\frac{T}{2}), X_7(t), X_7(t+\frac{T}{2}))$. Alternation occurs in each column but may at different time. As a result, all perceptions may occur with similar probabilities.

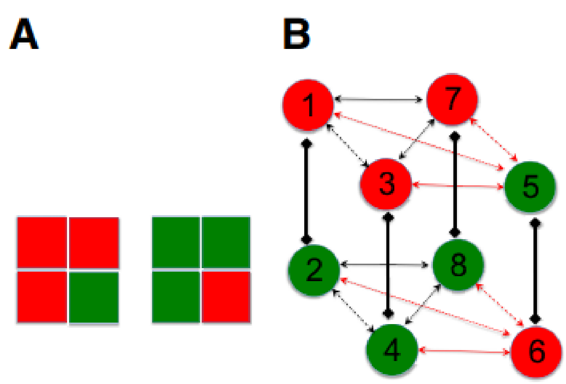


Figure 3.4.2. 4-location Binocular Rivalry: **A**-8 rectangles in red and in green of identical size, arranged in configuration H; **B**: Wilson network with 4 attributes and 2 levels. (Golubitsky et al., 2019, p. 1991).

3.5 Binocular Rivalry Experiment Analysis

The experiment introduced in this subsection is designed to obtain empirical data to verify the analytical results produced by the previous 4-location binocular model.

3.5.1 Experiment Design

The experiment was proposed and conducted by Golubitsky (2019), three observers participated in the experiment. They are presented by images in the configurations with a stereoscope. Configuration A and B has no spacing. in configuration C-H, black lines were presented around each colored square. Each observer was trained previously before the main sessions. There are three main sessions. Each of them consists of eight 5-minute block (one for each configuration). In each 5-min block, observers were presented with the configuration color images for 2s (this duration is based on a pilot experiment where observers typically reported binocular rivalry after observing for 2s). After this presentation, the were asked to report what they have seen with arrow keys on the keyboard.

3.5.2 Experiment Results

The experiment results are presented in the tables below.

Configuration	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
RRRR	59*	33	33*	59	15	20	42	21
RGRR	0	0	15	9	12	19	33	4
RRGR	0	0	2	4	4	5	1	15*
RGGR	1	0	1	1	24*	12	3	7
RRRG	1	2	9	4	10	12	6	14
RGRG	0	0	5	1	12	20*	20	5
RRGG	0	16*	10	10*	0	2	2	1
RGGG	0	0	4	1	5	1	16†	1
GRRR	0	15	18	50	22	40	27*	22
GGRR	13	138†	32	68†	10	14	7	20
GRGR	0	1	0	6	16	31†	2	11
GGGR	0	1	4	4	9	11	1	12
GRRG	58	1	47	21	104†	60	51	62†
GGRG	16	1	55	6	10	11	16	60
GRGG	1	10	12	12	8	3	20	5
GGGG	122†	37	19†	11	3	1	17	2

Table 3.5.2.1. Experiment results reported by observer 1. (Golubitsky et al., 2019, p. 1996).

Configuration	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
RRRR	135*	33	110*	66	44	58	47	53
RGRR	0	0	3	9	4	11	5	2
RRGR	0	0	2	14	10	3	2	28*
RGGR	0	2	7	7	48*	6	2	19
RRRG	0	0	7	4	9	6	13	4
RGRG	0	0	2	7	7	12*	5	0
RRGG	1	10*	2	17*	3	3	11	2
RGGG	0	1	0	13	8	2	76†	2
GRRR	0	0	7	13	3	20	21*	0
GGRR	0	121†	4	23†	0	2	0	1
GRGR	0	0	1	14	4	34†	3	7
GGGR	0	7	1	12	5	16	2	8
GRRG	0	1	30	3	54†	11	21	9
GGRG	0	1	5	9	8	8	2	55†
GRGG	1	0	6	12	8	24	10	6
GGGG	122†	64	60†	17	27	17	22	46

Table 3.5.2.2. Experiment results reported by observer 2. (Golubitsky et al., 2019, p. 1996).

Configuration	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
RRRR	108*	61	48*	21	26	10	10	39
RGRR	0	3	12	10	7	10	19	9
RRGR	0	2	3	0	6	6	1	26*
RGGR	1	0	9	0	46*	6	24	14
RRRG	0	2	1	8	1	6	1	2
RGRG	0	4	2	6	5	23*	4	1
RRGG	1	27*	1	48*	5	4	3	4
RGGG	0	0	1	11	6	18	26†	2
GRRR	0	10	2	0	4	8	35*	2
GGRR	1	59†	50	9†	4	1	32	8
GRGR	0	4	3	0	8	3†	16	10
GGGR	1	7	29	3	37	2	4	39
GRRG	0	0	0	3	10†	39	13	1
GGRG	0	3	7	1	3	27	1	24†
GRGG	1	3	1	16	28	21	20	5
GGGG	134†	28	65†	105	29	49	33	46

Table 3.5.2.3. Experiment results reported by observer 3. (Golubitsky et al., 2019, p. 1996).

For configuration A, only pure color perceptions was reported by observer 2 and 3, while observer 1 also reported $GGRR, GRRG$, and $GGRG$ perceptions. With spacing (see experiment results for configration C and compare it with A), observer 2 and 3 also reported perception other that pure color ($RRRR, GGGG$), which give evidence that spacing affects perceptions (which is not considered in the model above). From the results we can also notice that the probability of having even perceptions (perceptions that have even numbers of green or red color squares. e.g., $GGRR$) are highest in non-spacing pure color configurations, then even configurations (C-F), and lowest in odd configuration.

The experiment results agrees with analytical results produced by previous model analysis in the sense of frequency. However, the predictions of the model does not agree with all results in the experiment. The spacing pure color experiment, for example, produces results other than pure color perceptions, which disagrees with the prediction.

3.6 Generalized Rivalry Network

In the 8-node (4-attribute 2-level) binocular rivalry model, it is assumed that the connection between nodes with different colors, different spatial positions (e.g. horizontal and vertical) are different. This assumption reduced the numbers of symmetry operations that can be applied to the model. For example, without considering the color feature, both τ and κ in the previous subsection can be applied to configuration G and H. With removal of assumptions, more symmetry operations can be added and the model can be modified to analyze other rivalry problems. The 8-node generalized rivalry that is used to produce quadrupedal gaits predictions, is a rivalry network consists of symmetry group $Z_4 \times Z_2$.

4 Conclusion and Discussion

In conclusion, the rivalry network in the context of visual perception gives explanations of the binocular rivalry phenomena in 1 and 2 retinal cases. With the involvement of symmetry, we can see the potential of extending the model to higher dimension and explaining more complicated visual illusion (e.g. the Necker cube illusion, which describes the alternation of perceptions for a same cube (Stewart & Golubitsky, 2019)). With current mathematical approach, our analyses focus on giving "the menu of possible perceptions" predicted by symmetries of the generalized rivalry network. However, this method is based on the assumptions about symmetry, which, in the current stage is not grounded by empirical evidences but assumptions about the properties of connections, the subdivision hypothesis (which assume images, like the monkey-text mixed image, can be subdivided into pieces of perceptually equivalent pieces), and the model's invariance of other elements in the image (e.g. the spacing in the previous experiment). This can be problematic even the rivalry network can predict the exact results of many experiment, as we can see disagreement between the predicted results and the previous experiment results. Moreover, there is no experiment that directly tests the reliability of the model. In this sense, future studies may consider developing a systematic experimental approach that obtained detailed dynamics of binocular rivalry and directly test the properties of the periodical solutions (Golubitsky, 2019, p. 1997).

5 Appendix

5.1 Listing 1: Mathmatica Code for Wilson-Cowan Model Simulation

```
a = 0.1
d = 0.1
b = 1
c = 1
w = 1
Manipulate[
 Show[ StreamPlot[{-a x + ( w x - b y + i)/
      Sqrt[(w x - b y + i)^2 + 1], -d y + (c x + j)/
      Sqrt[(c x + j)^2 + 1]}, {x, -20, 20}, {y, -20, 20},
   StreamColorFunction -> "Rainbow"],
  ContourPlot[-d y + (c x + j)/Sqrt[(c x + j)^2 + 1] == 0, {x, -20,
    20}, {y, -20, 20}, ColorFunction -> Yellow],
  ContourPlot[-a x + (w x - b y + i)/Sqrt[(w x - b y + i)^2 + 1] ==
    0, {x, -20, 20}, {y, -20, 20}, ColorFunction -> Yellow],
  ParametricPlot[Evaluate[First[{x[t], y[t]} /.

      NDSolve[{x'[
          t] == -a x[t] + ( w x[t] - b y[t] + i)/
           Sqrt[(w x[t] - b y[t] + i)^2 + 1],
         y'[t] == -d y[t] + (c x[t] + j)/Sqrt[(c x[t] + j)^2 + 1],
         Thread[{x[0], y[0]} == point]}, {x, y}, {t, 0, 1000}]]], {t,
    0, 1000}, PlotStyle -> Red]], {{point, {0.02, 0}},
  Locator}, {i, -5, 5}, {j, -10, 10}, SaveDefinitions -> True]

```

Listing 1: 2-node Wilson-Cowan Model Simulation-Mathematica

5.2 Explanation 1: Symmetry

Definition. *Symmetry* is an operation that transform the original arrangements of nodes to a state that is indistinguishable from the original state.

An intuitive explanation of "indistinguishable" is that the connections between nodes are identical with the original arrangement. For example, assume all connections are of the same type, then after the operation the approximation of the graph should be identical with the original graph. The simplest example of a symmetry operation is shown below in Graph 5.2.1, where $A(12)$ is a symmetry operation.

Figure 5.2.1. A symmetry operation

Definition. Symmetry group is a group of symmetry operation, denoted as $\mathbf{Z}_n$, where n is the number of nodes that the symmetry operation in the symmetry group operates on.

For Z_n to be a group, it must satisfy the group axiom:

- **Associativity**: For all $a, b, c \in Z_n$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Existence of identity element**: There exists an element e in Z_n such that $e \cdot a = a$ and $a \cdot e = a$.
- **Existence of inverse element**: For each $a \in Z_n$, there exists an unique b such that $a \cdot b = e$ and $b \cdot a = e$. b is denoted as a^{-1}

Following the example above, we can quickly check that the symmetry group $Z_2(A)$, which contains only one symmetry operation A, is a group:

- **Associativity**: $(A \cdot A) \cdot A = A \cdot (a \cdot A)$ as they both end up flipping the nodes
- **Existence of identity element**: A trivial bijection serves $I = x \mapsto x$ the element.
- Existence of inverse element: $A \cdot A = I$, $A = A^{-1}$.

Remark. Z_n is abel if and only if if is commutative: For each $a, b \in Z_n$, $a \cdot b = b \cdot a$. Not all symmetry group are Abel (Counterexample in graph 5.2.2). The symmetry groups discussed in this paper are all abel.

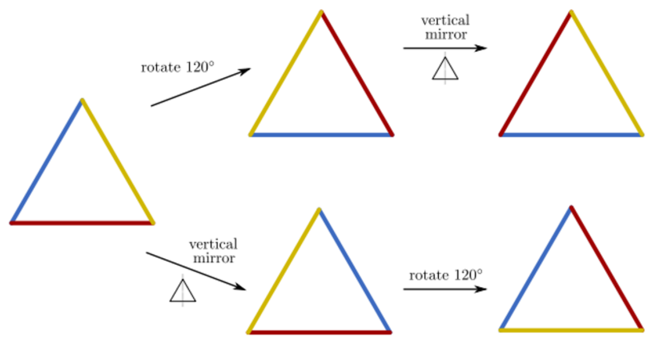


Figure 5.2.2. A Symmetry group Z_3(*rotate 120°*, *mirror*) is not abel

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